

\mathbb{Z}_2 -COEFFICIENT HOMOLOGY $(1, 2)$ -SYSTOLIC FREEDOM OF $\mathbb{R}P^3 \# \mathbb{R}P^3$

LIZHI CHEN

ABSTRACT. We prove \mathbb{Z}_2 -coefficient homology $(1, 2)$ -systolic freedom on the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$ of two copies of real projective 3-space. Given a Riemannian metric on $\mathbb{R}P^3 \# \mathbb{R}P^3$, we define the \mathbb{Z}_2 -coefficient homology 1-systole as the infimum length of all nonseparating geodesic loops representing nonzero classes in the first homology group with \mathbb{Z}_2 -coefficients. The \mathbb{Z}_2 -coefficient homology 2-systole is defined as the infimum area of all nonseparating surfaces which represent nonzero classes in the second homology group with \mathbb{Z}_2 -coefficients. In the paper we show that there exists a sequence of Riemannian metrics on $\mathbb{R}P^3 \# \mathbb{R}P^3$ such that the volume cannot be bounded below in terms of the product of \mathbb{Z}_2 -coefficient homology 1-systole and \mathbb{Z}_2 -coefficient homology 2-systole.

CONTENTS

1. Introduction	1
2. 3-manifolds with Surface Bundle Structure	5
3. Freedman's Example of $S^2 \times S^1$	7
4. 3-manifolds with Semibundle Structure	10
5. Proof of Main Theorem	12
5.1. Construction of Riemannian Metrics on $\mathbb{R}P^3 \# \mathbb{R}P^3$	12
5.2. \mathbb{Z}_2 -coefficient Homology $(1, 2)$ -systolic Freedom	15
References	18

1. INTRODUCTION

In this paper, we investigate the problem of whether the Riemannian volume of a manifold can be bounded below in terms of representatives of homotopy or homology classes. In literature, the existence of such

Date: February 20, 2014.

2010 Mathematics Subject Classification. Primary 53C23.

lower bounds is expressed by the so called systolic inequality. The violation of systolic inequality is called as systolic freedom. The research of systolic inequality is initiated by C. Loewner's work on torus. After C. Loewner, P. Pu proved another systolic inequality on real projective plane.

Let \mathbb{T}^2 be a smooth torus with the Riemannian metric \mathcal{G} , denoted by $(\mathbb{T}^2, \mathcal{G})$. We use $\text{Area}_{\mathcal{G}}(\mathbb{T}^2)$ to denote the area of \mathbb{T}^2 under the Riemannian metric \mathcal{G} . We define the shortest length of all noncontractible loops in \mathbb{T}^2 as the homotopy 1-systole of $(\mathbb{T}^2, \mathcal{G})$, denoted by $\text{Sys } \pi_1(\mathbb{T}^2, \mathcal{G})$. In 1949, C. Loewner proved that every Riemannian metric \mathcal{G} on \mathbb{T}^2 satisfies the inequality

$$\text{Sys } \pi_1(\mathbb{T}^2, \mathcal{G})^2 \leq \frac{2}{\sqrt{3}} \text{Area}_{\mathcal{G}}(\mathbb{T}^2), \quad (1.1)$$

see Chapter 5 of [16]. The equality in (1.1) holds on a flat hexagonal torus, i.e., a torus with flat Riemannian metric defined by \mathbb{R}^2/Λ , where Λ is the lattice in \mathbb{R}^2 spanned by vectors $(1, 0)$ and $(1/2, \sqrt{3}/2)$.

After C. Loewner, P. Pu proved a same type inequality on real projective plane. Let $\mathbb{R}P^2$ be a smooth real projective plane with the Riemannian metric \mathcal{G} , denoted by the pair $(\mathbb{R}P^2, \mathcal{G})$. Similarly as above, we use $\text{Area}_{\mathcal{G}}(\mathbb{R}P^2)$ to denote the area of $\mathbb{R}P^2$. And the homotopy 1-systole of $(\mathbb{R}P^2, \mathcal{G})$, denoted by $\text{Sys } \pi_1(\mathbb{R}P^2, \mathcal{G})$, is the shortest length of all noncontractible loops in $\mathbb{R}P^2$. P. Pu showed that every Riemannian metric \mathcal{G} on $\mathbb{R}P^2$ satisfies the inequality

$$\text{Sys } \pi_1(\mathbb{R}P^2, \mathcal{G})^2 \leq \frac{\pi}{2} \text{Area}_{\mathcal{G}}(\mathbb{R}P^2), \quad (1.2)$$

where equality holds for metrics \mathcal{G} with constant Gaussian curvature, see [16] or [20].

Motivated by Loewner inequality (1.1) and Pu inequality (1.2), we investigate whether the area of a general surface or even the volume a higher dimensional Riemannian manifold can be bounded below in terms of its homotopy 1-systole. Let M be a Riemannian manifold with the Riemannian metric \mathcal{G} , denoted by (M, \mathcal{G}) . For a noncontractible geodesic loop $\gamma \in M$, we use $\text{length}_{\mathcal{G}}(\gamma)$ to denote the length of γ with respect to the metric \mathcal{G} .

Definition 1.1. The homotopy 1-systole of (M, \mathcal{G}) , denoted by

$$\text{Sys } \pi_1(M, \mathcal{G}),$$

is defined as

$$\inf_{\gamma} \text{length}_{\mathcal{G}}(\gamma),$$

where the infimum is over all noncontractible loops γ in M .

On closed surfaces, we have the following theorem of systolic inequality.

Theorem 1.2 (Gromov, Corollary 5.2.B. of [9]). *Let Σ be a closed surface which is not homeomorphic to a 2-sphere or a real projective plane, then every Riemannian metric \mathcal{G} on Σ satisfies*

$$Sys \pi_1(\Sigma, \mathcal{G})^2 \leq \frac{4}{3} Area_{\mathcal{G}}(\Sigma). \quad (1.3)$$

For manifolds with dimension $n \geq 3$, M. Gromov defined the notion of essential manifolds. An aspherical space K is a topological space with all higher homotopy groups $\pi_i(K)$ vanishing, where $i \geq 2$. Let M be a compact and orientable manifold of dimension n , with $n \geq 3$. The manifold M is essential if there exists a map $f : M \rightarrow K$ from M to an aspherical space K , such that the induced homomorphism $f_* : H_n(M; \mathbb{Z}) \rightarrow H_n(K; \mathbb{Z})$ maps the fundamental class $[M]$ to a nonzero class in $H_n(K; \mathbb{Z})$. When M is nonorientable, in the definition we let $[M] \in H_n(M; \mathbb{Z}_2)$. M. Gromov proved the following theorem for essential Riemannian manifolds.

Theorem 1.3 (Gromov, Theorem 0.1.A. of [9]). *Let M be a compact essential manifold with dimension n , where $n \geq 3$. Every Riemannian metric \mathcal{G} on M satisfies*

$$Sys \pi_1(M, \mathcal{G})^n \leq C(n) Vol_{\mathcal{G}}(M), \quad (1.4)$$

where $C(n) = \left(6(n+1)n^n \sqrt{(n+1)!}\right)^n$ is a positive constant which only depends on the dimension n .

A generalization of homotopy 1-systole is to consider the infimum volume of cycles representing nonzero classes in homology groups. Let M be a manifold of dimension n , where $n \geq 3$. Let \mathcal{G} be a Riemannian metric on M . Let Δ^k be the standard k -simplex in \mathbb{R}^n , where $1 \leq k \leq n-1$. We define the volume of a Lipschitz k -simplex $\sigma_k : \Delta^k \rightarrow M$ as the integral

$$\int_{\Delta^k} dV_{\sigma^*(\mathcal{G})},$$

where $dV_{\sigma^*(\mathcal{G})}$ is the volume form of the pullback metric $\sigma^*(\mathcal{G})$. We use $Vol_{\mathcal{G}}(\sigma_k)$ to denote the volume of σ_k . For a cycle $c = \sum_i a_i \sigma_i$ with coefficients $a_i \in \mathbb{Z}$ or \mathbb{Z}_2 , the volume of c is defined as $\sum_i |a_i| Vol_{\mathcal{G}}(\sigma_i)$, denoted by $Vol_{\mathcal{G}}(c)$. Then we have following definition of norms on $H_k(M; \mathbb{Z})$ and $H_k(M; \mathbb{Z}_2)$.

Definition 1.4. Let $\alpha \in H_k(M; \mathbb{Z})$. The norm of α is defined as

$$\|\alpha\| = \inf_c Vol_{\mathcal{G}}(c),$$

where the infimum is over all cycles c representing α .

The norm of a class $\beta \in H_k(M; \mathbb{Z}_2)$ is defined in the same way. In terms of norms defined on homology groups, we define the homology k -systole as follows.

Definition 1.5. The \mathbb{Z} -coefficient homology k -systole of a Riemannian manifold (M, \mathcal{G}) is defined as

$$\inf_{\alpha \in H_k(M; \mathbb{Z}) \setminus \{0\}} \|\alpha\|,$$

where the infimum is over all nonzero classes in $H_k(M; \mathbb{Z})$. The \mathbb{Z}_2 -coefficient homology k -systole of (M, \mathcal{G}) is defined as

$$\inf_{\beta \in H_k(M; \mathbb{Z}_2) \setminus \{0\}} \|\beta\|,$$

where the infimum is over all nonzero classes β in $H_k(M; \mathbb{Z}_2)$.

Different from the homotopy 1-systole, violations of systolic inequality exist for higher homology k -systoles. We call such a phenomenon as systolic freedom. In particular, we have the following definition of systolic freedom.

Definition 1.6. Let M be a manifold of dimension n , where $n \geq 3$. Then M has \mathbb{Z} -coefficient homology $(1, n-1)$ -systolic freedom if we have

$$\inf_{\mathcal{G}} \frac{\text{Vol}_{\mathcal{G}}(M)}{\text{Sys } H_1(M, \mathcal{G}; \mathbb{Z}) \cdot \text{Sys } H_{n-1}(M, \mathcal{G}; \mathbb{Z})} = 0,$$

where the infimum is over all Riemannian metrics \mathcal{G} on M .

The \mathbb{Z}_2 -coefficient homology $(1, n-1)$ -systolic freedom is defined in the same way. In [4], L. Bergery and M. Katz proved the following theorem.

Theorem 1.7 (L. Bergery and M. Katz, [4]). *The 3-manifold $S^2 \times S^1$ has \mathbb{Z} -coefficient homology $(1, 2)$ -systolic freedom.*

Moreover, I. Babenko and M. Katz proved the following result of systolic freedom in [2].

Theorem 1.8 (Babenko and Katz, [2]). *Let M be a compact and orientable manifold of dimension n , where $n \geq 3$. Then the manifold M has \mathbb{Z} -coefficient homology $(1, n-1)$ -systolic freedom.*

The systolic freedom of \mathbb{Z} -coefficient homology k -systoles generally exist. More results can be found in [2], [3], [14], [15], [17], [18] and [19].

M. Gromov has conjectured that the systolic inequality exists for \mathbb{Z}_2 -coefficient homology k -systoles, see [7]. However, in 1999 M. Freedman constructed a counterexample in [7]. He proved the \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom on $S^2 \times S^1$.

Theorem 1.9 (Freedman, [7]). *The 3-manifold $S^2 \times S^1$ has \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom.*

We will review M. Freedman's example in Section 2 of this paper.

M. Freedman's example is the first result of \mathbb{Z}_2 -coefficient homology k -systoles for manifolds with dimension at least 3. In this paper, we show a further development based on M. Freedman's work of $S^2 \times S^1$. Let $\mathbb{R}P^3\#\mathbb{R}P^3$ be the 3-manifold which is the connected sum of two copies of real projective 3-space. We prove the 3-manifold $\mathbb{R}P^3\#\mathbb{R}P^3$ exhibits \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom. The main theorem of the paper is as follows.

Theorem 1.10. *The 3-manifold $\mathbb{R}P^3\#\mathbb{R}P^3$ has \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom.*

The paper is organized as follows. In Section 2, we introduce some geometric properties of 3-manifolds with surface bundle structure, as well as Dehn surgery of 3-manifolds. In Section 3, we review M. Freedman's example of \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom on $S^2 \times S^1$. In Section 4, we introduce the semibundle structure of 3-manifolds. In Section 5, we prove the main theorem 1.10 of this paper.

2. 3-MANIFOLDS WITH SURFACE BUNDLE STRUCTURE

In this section, we introduce some preliminary knowledge of 3-manifolds with surface bundle structure, i.e., 3-manifolds which fiber over the circle. The class of 3-manifolds with surface bundle structure is very important in 3-manifold theory, see [1] and [23]. W. Thurston's theory establishes a geometric classification for 3-manifolds in this class. The 3-manifold which fibers over circle has a mapping torus structure. The theorem of W. Thurston is based on the classification of the monodromy of mapping torus. We introduce the mapping class group of surfaces first. The definition of Dehn twist will be given. The Dehn-Lickorish theorem implies that the mapping class group of surfaces is generated by Dehn twists, which is a fundamental theorem in mapping class group theory. Moreover, we have a short introduction to the Dehn surgery of 3-manifolds.

Let S be a connected and closed surface. In this paper, we use the convention that a surface is closed if it is compact and without boundary. We use $\text{Diff}^+(S)$ to denote the group of orientation preserving

diffeomorphisms. And $\text{Diff}_0(S)$ stands for the subgroup of $\text{Diff}^+(S)$ which consists of elements isotopic to the identity.

Definition 2.1. The mapping class group of S , denoted by $\text{Mod}(S)$, is defined to be the group of isotopy classes of orientation preserving diffeomorphisms of S :

$$\text{Mod}(S) = \text{Diff}^+(S)/\text{Diff}_0(S).$$

An element in $\text{Mod}(S)$ is called as a mapping class. The Dehn-Lickorish theorem shows that the mapping class group of an oriented surface is generated by Dehn twists. Let A be the annulus $S^1 \times [0, 1]$, which is represented by

$$\{(e^{i\theta}, t) \mid 0 \leq \theta \leq 2\pi, 0 \leq t \leq 1\}.$$

Or equivalently, we represent A by the set $\{(\theta, t) \mid 0 \leq \theta \leq 2\pi, 0 \leq t \leq 1\}$. A twisting map $\psi : A \rightarrow A$ is defined as

$$(\theta, t) \mapsto (\theta + 2\pi t, t).$$

Let γ be a simple loop on closed surface S , with a neighborhood $\mathcal{C}(\gamma)$ in S . There exists a diffeomorphism $H : \mathcal{C}(\gamma) \rightarrow A$.

Definition 2.2. The Dehn twist on γ is defined to be the diffeomorphism $H^{-1} \circ \psi \circ H : \mathcal{C}(\gamma) \rightarrow \mathcal{C}(\gamma)$.

Let Σ_g be a closed surface with genus g . We have the following Dehn-Lickorish theorem.

Theorem 2.3 (Theorem 4.1, [6]). *The mapping class group $\text{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists along nonseparating simple loops in Σ_g .*

Remark 2.4. Proved by Humphries, the number of nonseparating simple loops in Dehn-Lickorish theorem can be taken as $2g + 1$, see [6].

Moreover, elements in $\text{Mod}(\Sigma_g)$ are classified into three types: periodic, reducible and pseudo-Anosov, see 13.3 of [6]. An element $f \in \text{Mod}(\Sigma_g)$ is periodic if it has finite order. For a periodic mapping class $f \in \text{Mod}(\Sigma_g)$, there exists a representative $\phi \in \text{Diff}^+(\Sigma_g)$ so that ϕ has finite order, see Theorem 7.1 in [6].

Let M be a smooth manifold, and $\phi : M \rightarrow M$ is a diffeomorphism.

Definition 2.5. The mapping torus M_ϕ is a fiber bundle over the map ϕ , with fiber the manifold M , which is obtained from the cylinder $M \times [0, 1]$ by identifying the two ends via the map ϕ .

A 3-manifold with surface bundle structure is a fiber bundle over the circle S^1 , with the fiber a closed orientable surface. Let M be a 3-manifold with surface bundle structure, then it has the mapping torus structure:

$$M = \Sigma_g \times [0, 1] / (x, 0) \sim (\phi(x), 1),$$

where Σ_g is the fiber surface of M , and the mapping class represented by $\phi : \Sigma_g \rightarrow \Sigma_g$ is called as the monodromy of M . According to W. Thurston, the geometric structure of M is dependent on the type of the monodromy ϕ . Let \mathbb{H}^2 be the hyperbolic plane and \mathbb{R} be the Euclidean line. The following proposition is contained in W. Thurston's theorem.

Proposition 2.6 (see Theorem 13.4, [6]). *Let M be a 3-manifold with surface bundle structure. Then M has $\mathbb{H}^2 \times \mathbb{R}$ geometric structure if and only if the monodromy of M is periodic.*

At last, we introduce Dehn surgery of 3-manifolds. Let M be a 3-manifold, with boundary possibly. Let K be a knot in the interior of M . We use $\mathcal{T}(K)$ to denote a tubular neighborhood of γ in the interior of M , which is a solid torus. Let γ be a loop on the boundary torus $\partial\mathcal{T}(K)$.

Definition 2.7. The Dehn surgery on M with respect to the knot K and the loop γ is the following construction:

$$M' = (M - (\mathcal{T}(K))^\circ) \cup_f \tilde{\mathcal{T}},$$

where $\tilde{\mathcal{T}}$ is a solid torus, and $f : \partial\mathcal{T}(K) \rightarrow \partial\tilde{\mathcal{T}}$ is a homeomorphism on the boundary tori, such that the meridian loop of $\partial\tilde{\mathcal{T}}$ is mapped onto the loop γ on $\partial\mathcal{T}(K)$.

We also call the operation of gluing a solid torus $\tilde{\mathcal{T}}$ to the 3-manifold $M - (\mathcal{T}(K))^\circ$ with torus boundary as Dehn filling.

3. FREEDMAN'S EXAMPLE OF $S^2 \times S^1$

In [7] (see also [8]), M. Freedman proved the \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom of $S^2 \times S^1$. We introduce the technique and outline of his proof in this section. In the proof, a sequence of arithmetic hyperbolic surfaces $\{\Sigma_{g_k}\}_{k=1}^\infty$ is constructed. Then based on the hyperbolic surface Σ_{g_k} , a Riemannian mapping torus M_{g_k} with the metric \mathcal{G}_k is constructed, where the Riemannian metric \mathcal{G}_k is locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. The mapping torus M_{g_k} has fiber surface Σ_{g_k} . Then a sequence of Dehn surgeries is performed to convert M_{g_k} into a 3-manifold homeomorphic to $S^2 \times S^1$, denoted by $S^2 \times S_{g_k}^1$. After

specifying the metric change in Dehn surgeries, we have a Riemannian metric $\widehat{\mathcal{G}}_k$ on $S^2 \times S^1_{g_k}$. The metric change in Dehn surgeries is not given in M. Freedman's papers [7] and [8]. We will describe such metric change in Section 5 of this article. Based on topological and geometrical properties of M_{g_k} as well as the metric change in Dehn surgeries, M. Freedman estimated the growth of $\text{Sys } H_1(S^2 \times S^1, \widehat{\mathcal{G}}_k; \mathbb{Z}_2)$ and $\text{Sys } H_2(S^2 \times S^1, \widehat{\mathcal{G}}_k; \mathbb{Z}_2)$ as well as the volume $\text{Vol}_{\widehat{\mathcal{G}}_k}(S^2 \times S^1)$. These estimations yield that

$$\inf_k \frac{\text{Vol}_{\widehat{\mathcal{G}}_k}(S^2 \times S^1)}{\text{Sys } H_1(S^2 \times S^1, \widehat{\mathcal{G}}_k; \mathbb{Z}_2) \cdot \text{Sys } H_2(S^2 \times S^1, \widehat{\mathcal{G}}_k; \mathbb{Z}_2)} = 0. \quad (3.1)$$

Hence the 3-manifold $S^2 \times S^1$ has \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom. In the following we show more details of M. Freedman's work.

Let p be a prime number such that $p \equiv 3 \pmod{4}$. We define the group $\Gamma_{(-1, p)}$ as

$$\left\{ \begin{pmatrix} a + b\sqrt{p} & -c + d\sqrt{p} \\ c + d\sqrt{p} & a - b\sqrt{p} \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \det = 1 \right\} / \pm \text{I}_2,$$

where \det denotes the determinant of the 2×2 matrix in the group. The group $\Gamma_{(-1, p)}$ is an arithmetic Fuchsian group derived from the quaternion algebra

$$\left(\frac{-1, p}{\mathbb{Q}} \right),$$

see [22]. Let $N \geq 2$ be a positive integer. We define the N -th congruence subgroup of $\Gamma_{(-1, p)}$ as

$$\left\{ \begin{pmatrix} 1 + N(a + b\sqrt{p}) & N(-c + d\sqrt{p}) \\ N(c + d\sqrt{p}) & 1 + N(a - b\sqrt{p}) \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \det = 1 \right\} / \pm \text{I}_2,$$

denoted by $\Gamma_{(-1, p)}(N)$. By using the congruence subgroup $\Gamma_{(-1, p)}(N)$, we construct the arithmetic Riemann surface $\mathbb{H}^2 / \Gamma_{(-1, p)}(N)$. We have the following properties of the arithmetic Riemann surface $\mathbb{H}^2 / \Gamma_{(-1, p)}(N)$, see [7] and [22].

Proposition 3.1 (P. Schmutz Schaller, [22]). (1) *The arithmetic Riemann surface $\mathbb{H}^2 / \Gamma_{(-1, p)}(N)$ is hyperbolic.*

(2) *If we use $\text{genus}(N)$ to denote the genus of $\mathbb{H}^2 / \Gamma_{(-1, p)}(N)$, then we have*

$$A_p N^2 \leq \text{genus}(N) \leq B_p N^3,$$

where A_p and B_p are two positive constants which only depend on p .

(3) Let $\mathcal{G}_{\mathbb{H}^2}$ be the hyperbolic metric on $\mathbb{H}^2/\Gamma_{(-1,p)}(N)$, then we have

$$\text{Sys } \pi_1(\mathbb{H}^2/\Gamma_{(-1,p)}(N), \mathcal{G}_{\mathbb{H}^2}) \geq C \log N,$$

where C is a fixed positive constant independent of N .

In the following, we use Σ_g to denote the arithmetic hyperbolic surface $\mathbb{H}^2/\Gamma_{(-1,p)}(N)$ with genus g . From the above proposition, we have

$$\text{Sys } \pi_1(\Sigma_g, \mathcal{G}_{\mathbb{H}^2}) \geq C' \log g, \quad (3.2)$$

where C' is a fixed positive constant number independent of g . Moreover, by P. Buser and P. Sarnak [5], we have

$$\lambda_1(\Sigma_g) \geq c_1, \quad (3.3)$$

where λ_1 is the first eigenvalue of the Laplacian on Σ_g , and c_1 is a fixed positive constant number independent of g . Let g be large enough, we can construct an isometry map $\tau : \Sigma_g \rightarrow \Sigma_g$ of finite order (see [7]), such that

$$\text{Order}(\tau) \geq c_2 (\log g)^{1/2},$$

where $\text{Order}(\tau)$ stands for the order of τ , and c_2 is a fixed positive constant independent of g . Therefore, we have a sequence of arithmetic hyperbolic surfaces $\{\Sigma_{g_k}\}_{k=1}^\infty$ such that

$$2 \leq g_1 < g_2 < \cdots < g_k < \cdots \quad \text{and} \quad \lim_{k \rightarrow \infty} g_k = \infty,$$

with a finite order isometry $\tau_k : \Sigma_{g_k} \rightarrow \Sigma_{g_k}$ associated with each Σ_{g_k} .

Let the arithmetic hyperbolic surface Σ_{g_k} be the fiber surface, we construct a mapping torus

$$M_{g_k} = \Sigma_{g_k} \times [0, 1] / (x, 0) \sim (\tau_k(x), 1).$$

As the monodromy represented by τ_k has finite order, by Thurston's theorem (Proposition 2.6), M_{g_k} has geometric structure $\mathbb{H}^2 \times \mathbb{R}$. Then on M_{g_k} we have a Riemannian metric \mathcal{G}_k which is locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. In terms of the sequence of arithmetic hyperbolic surfaces $\{\Sigma_{g_k}\}_{k=1}^\infty$, we have a sequence of Riemannian mapping tori $\{(M_{g_k}, \mathcal{G}_k)\}_{k=1}^\infty$.

As the isometry map τ_k has finite order, by Dehn-Lickorish theorem (Theorem 2.3), we have

$$\tau_k = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n_k},$$

where $\sigma_1, \sigma_2, \dots, \sigma_{n_k}$ are Dehn twists along $2g_k + 1$ nonseparating simple geodesic loops $\gamma_1, \gamma_2, \dots, \gamma_{2g_k+1}$ in Σ_{g_k} . On M_{g_k} we perform a Dehn surgery to finish each Dehn twist σ_i . Then we have a mapping torus with monodromy represented by $\tau_k^{-1} \circ \tau_k$, which is the identity. Hence we obtain a mapping torus M'_{g_k} , which is homeomorphic to $\Sigma_{g_k} \times S^1$.

The arithmetic hyperbolic surface Σ_{g_k} has a system of $2g_k$ nonseparating geodesic loops representing a basis of $H_1(\Sigma_{g_k}; \mathbb{Z})$. Around each geodesic loop, we perform a Dehn surgery to let the geodesic loop be contractible. Then we have a 3-manifold M''_{g_k} homeomorphic to $S^2 \times S^1$, denoted by $S^2 \times S^1_{g_k}$. In all Dehn surgeries, we define the Riemannian metric on filled in solid tori as the Euclidean one. Then we use a cutoff function to get a smooth Riemannian metric after each Dehn surgery, see Section 5. Finally after all $n_k + 2g_k$ Dehn surgeries, we obtain a smooth Riemannian metric $\hat{\mathcal{G}}_k$ on $S^2 \times S^1_{g_k}$. Hence we get a sequence of Riemannian metrics on $S^2 \times S^1$, denoted by

$$\{(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k)\}_{k=1}^{\infty}.$$

In terms of the homotopy 1-systole estimation of Σ_{g_k} and metric changes in Dehn surgeries, we have the following estimations of \mathbb{Z}_2 -coefficient homology 1-systole and 2-systole, as well as the volume of $S^2 \times S^1_{g_k}$.

Theorem 3.2 (Freedman, [7]). (1) *There exists a positive constant c_4 independent of g_k such that*

$$\text{Sys } H_1(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq c_4 (\log g_k)^{1/2}. \quad (3.4)$$

(2) *There exists a positive constant c_5 independent of g_k such that*

$$\text{Sys } H_2(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq c_5 g_k. \quad (3.5)$$

(3) *There exists a positive constant c_6 independent of g_k such that*

$$\text{Vol}_{\hat{\mathcal{G}}_k}(S^2 \times S^1) \leq c_6 g_k. \quad (3.6)$$

Based on above estimations in the theorem, we obtain (3.1). Then we have the \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom on $S^2 \times S^1$.

4. 3-MANIFOLDS WITH SEMIBUNDLE STRUCTURE

Let M be a closed and connected 3-manifold. A halving of H is an index two subgroup of $\pi_1(M)$. When M has a halving H , there exists a two-sheeted covering 3-manifold whose fundamental group is isomorphic to H , denoted by M_H . Assume that $Q_H : M_H \rightarrow M$ is the covering map, and $\alpha_H : M_H \rightarrow M_H$ is the covering translation. Then we have $M \simeq M_H / \alpha_H$.

We express the unit circle S^1 as the set of complex numbers with unit module at the complex plane \mathbb{C}^1 . Let $D^1 = [-1, 1]$. Let $\tau : S^1 \rightarrow S^1$ be the map of complex conjugation, and $q : S^1 \rightarrow D^1$ be the projection map defined as $z \mapsto \text{Re}(z)$. Then we have the following definition of semibundle structure of 3-manifolds, see [24].

Definition 4.1. Let M be a closed and oriented connected 3-manifold with the halving H . A map $f : M \rightarrow D^1$ is called as a semibundle subordinate to the halving H (or an H -semibundle) with regular fiber Σ if there exists a two-sheeted covering surface bundle $F : M_H \rightarrow S^1$ with fiber surface Σ , such that $q \circ F = f \circ Q_H$ and $F \circ \alpha_H = \tau \circ F$.

The definition yields the following properties of an H -semibundle.

Proposition 4.2 (Zulli, [24]). *Let $f : M \rightarrow D^1$ be an H -semibundle. Let Σ_g be a closed surface with genus g . Assume that the covering surface bundle M_H has fiber surface Σ_g .*

- (1) *When $t \in (-1, 1)$, $f^{-1}(t)$ is homeomorphic to the fiber surface Σ_g of M_H . The regular fiber surface $f^{-1}(t)$ is lifted to two copies of Σ_g in M_H which are exchanged by α_H . When $t = -1$ or $t = 1$, $f^{-1}(t)$ is an embedded surface which is doubly covered by the fiber surface $F^{-1}(t) = \Sigma_g$.*
- (2) *If we use J to denote the interval $[-1, 0]$ or $[0, 1]$, then $f^{-1}(J)$ is a twisted I -bundle. We denote two twisted I -bundles $f^{-1}([-1, 0])$ and $f^{-1}([0, 1])$ by M_1 and M_2 respectively. Then we have $M = M_1 \cup M_2$, and $M_1 \cap M_2 = \Sigma_g$.*

We have an estimation for the homotopy 1-systole of an H -semibundle M .

Proposition 4.3. *Let M be an H -semibundle with the Riemannian metric \mathcal{G} . We use $\tilde{\mathcal{G}}$ to denote the covering metric on the covering surface bundle M_H . Then we have*

$$\text{Sys } \pi_1(M, \mathcal{G}) \geq \frac{1}{2} \text{Sys } \pi_1(M_H, \tilde{\mathcal{G}}). \quad (4.1)$$

Proof. For every noncontractible loop γ in M , there exists a lifting to the covering surface bundle M_H . We denote the lifting of γ as $\tilde{\gamma}$. Suppose that $[\gamma]$ is the homotopy class in $\pi_1(M)$ represented by γ , and $[\tilde{\gamma}]$ is the homotopy class in $\pi_1(M_H)$ represented by $\tilde{\gamma}$.

If $[\gamma] \in H$, as $(Q_H)_* : \pi_1(M_H) \rightarrow \pi_1(M)$ is injective and

$$(Q_H)_*(\pi_1(M_H)) = H,$$

we have $[\tilde{\gamma}] \neq 1$ in $\pi_1(M_H)$. Hence we have

$$\text{length}_{\mathcal{G}}(\gamma) = \text{length}_{\tilde{\mathcal{G}}}(\tilde{\gamma}).$$

If $[\gamma] \in \pi_1(M) - H$, as $\pi_1(M)/H \simeq \mathbb{Z}_2$, we have $\pi_1(M) = H \cup [\gamma]H$ and $[\gamma]^2 \in H$. Hence the union $\hat{\gamma} = \tilde{\gamma} \cup \alpha_H(\tilde{\gamma})$ is a noncontractible loop

in M_H , which doubly covers γ . Then we have

$$\text{length}_{\mathcal{G}}(\gamma) = \frac{1}{2} \text{length}_{\tilde{\mathcal{G}}}(\hat{\gamma}).$$

After taking infimum over all noncontractible loops γ in M , we have

$$\text{Sys } \pi_1(M, \mathcal{G}) \geq \frac{1}{2} \text{Sys } \pi_1(M_H, \tilde{\mathcal{G}}).$$

□

5. PROOF OF MAIN THEOREM

The proof of Theorem 1.10 is separated into two steps. In the first step, we apply M. Freedman's technique (see Section 3) to construct a sequence of Riemannian metrics on $\mathbb{R}P^3 \# \mathbb{R}P^3$, denoted by $\{(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \mathcal{G}_k)\}$. Then we prove the \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom in the second step.

5.1. Construction of Riemannian Metrics on $\mathbb{R}P^3 \# \mathbb{R}P^3$. Let $\{(\Sigma_{g_k}, \tau_k)\}_{k=1}^{\infty}$ be the sequence of arithmetic hyperbolic surfaces constructed in M. Freedman's example (see Section 3), where $\tau_k : \Sigma_{g_k} \rightarrow \Sigma_{g_k}$ is the isometry of finite order. We use I_1 to denote the interval $[-1, 0]$, and use I_2 to denote the interval $[0, 1]$. We define two twisted I -bundles over I_1 and I_2 with the regular fiber surface Σ_{g_k} , denoted by $\Sigma_{g_k} \tilde{\times} I_1$ and $\Sigma_{g_k} \tilde{\times} I_2$ respectively. Then for each k , we construct a 3-manifold N_{g_k} with the semibundle structure as

$$(\Sigma_{g_k} \tilde{\times} I_1) \cup_{\tau_k} (\Sigma_{g_k} \tilde{\times} I_2),$$

where the two twisted I -bundles are glued together along their common boundary Σ_{g_k} . The 3-manifold N_{g_k} is doubly covered by the mapping torus M_{g_k} (see Section 3). Then we have a Riemannian metric on N_{g_k} which is locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$, denoted by \mathcal{G}_k . The Riemannian metric \mathcal{G}_k on M_{g_k} is the covering metric. Moreover, if we let $H = \pi_1(M_{g_k})$, the 3-manifold N_{g_k} is an H -semibundle. Hence now we have a sequence of Riemannian H -semibundles $\{(N_{g_k}, \mathcal{G}_k)\}_{k=1}^{\infty}$.

As mentioned in Section 3, the isometry τ_k can be decomposed into a sequence of n_k number of Dehn twists along $2g_k + 1$ nonseparating simple geodesic loops on Σ_{g_k} . Around each geodesic loop, we perform a Dehn surgery. Moreover, all Dehn surgeries are performed at different levels of the regular fiber surface Σ_{g_k} . Thus they are pairwise disjoint when the radius ε_k of solid tori is small enough. After we finish all these Dehn surgeries, the 3-manifold N'_{g_k} obtained is homeomorphic to a semibundle which is doubly covered by the surface bundle $\Sigma_{g_k} \times S^1$. The metric change in Dehn surgeries is described as follows.

Suppose that $\tau_k = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n_k}$ (see Section 3), where $\sigma_1, \sigma_2, \dots, \sigma_{n_k}$ are Dehn twists along $2g_k + 1$ nonseparating simple geodesic loops on Σ_{g_k} . Moreover, we assume that the Dehn twist σ_i is along a nonseparating geodesic loop γ_i , for $i = 1, 2, \dots, n_k$. All Dehn surgeries in N_{g_k} corresponding to Dehn twists are performed at the following regular fiber surface levels:

$$\gamma_1 \times \left\{ \frac{1}{2} \right\}, \gamma_2 \times \left\{ \frac{1}{2} + \frac{1}{3n_k} \right\}, \dots, \gamma_{n_k} \times \left\{ \frac{1}{2} + \frac{n_k - 1}{3n_k} \right\}.$$

By the systolic property of the arithmetic hyperbolic surface Σ_{g_k} , we have

$$\text{length}_{\mathcal{G}_{\mathbb{H}^2}}(\gamma_i) \leq C \log g_k,$$

where C is a positive constant independent of g_k . We use $L_{i,k}$ to denote $\text{length}_{\mathcal{G}_{\mathbb{H}^2}}(\gamma_i)$ in the following. In Dehn surgery corresponding to the twist σ_i , denoted by \mathcal{D}_i , we first remove a solid torus T_{i,ε_k} with radius ε_k . The solid torus T_{i,ε_k} is a tubular neighborhood of a geodesic loop in N_{g_k} . And γ_i is the longitude loop in boundary torus $\partial T_{i,\varepsilon_k}$. We assume that the radius ε_k is equal to $1/g_k^2$. Let $\delta_k = \varepsilon_k/4$. We fill in a solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$ with radius $\varepsilon_k + \delta_k$ to the semibundle complement $N_{g_k} - T_{i,\varepsilon_k}^\circ$. The filled in solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$ is defined as follows. Let $\bar{T}_{i,\varepsilon_k+\delta_k}$ be the solid torus

$$\left\{ (r, \theta, t) \mid 0 \leq r \leq \frac{L_{i,k}}{2\pi} + \delta_k, 0 \leq \theta \leq 2\pi, 0 \leq t \leq 2\pi\varepsilon_k \right\} / \sim,$$

where \sim is the identification $(r, \theta, 0) \mapsto (r, \theta, 2\pi\varepsilon_k)$. On $\bar{T}_{i,\varepsilon_k+\delta_k}$, we define the Euclidean metric $\bar{\mathcal{G}}_k$, which is expressed as $dr^2 + r^2 d\theta^2 + dt^2$. Then we define a smooth twisting map $\beta_{i,k}$ on the solid torus $\bar{T}_{i,\varepsilon_k+\delta_k}$. When restricted to the disk

$$\{(r, \theta, \pi\varepsilon_k) \mid 0 \leq r \leq L_{i,k}, 0 \leq \theta \leq 2\pi\},$$

the twisting map $\beta_{i,k}$ is a π -rotation. While outside of this disk, it is the identity map. After performing the twisting $\beta_{i,k}$ on $\bar{T}_{i,\varepsilon_k+\delta_k}$, we have the desired filled in solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k} = \beta_{i,k}(\bar{T}_{i,\varepsilon_k+\delta_k})$. The metric on $\tilde{T}_{i,\varepsilon_k+\delta_k}$ is defined to be the pullback metric $(\beta_{i,k}^{-1})^* \bar{\mathcal{G}}_k$, denoted by $\tilde{\mathcal{G}}_k$.

Next we fill the solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$ to the semibundle complement $N_{g_k} - T_{i,\varepsilon_k}^\circ$. Let $\bar{Y}_{i,k}$ be an annulus product in the solid torus $\bar{T}_{i,\varepsilon_k+\delta_k}$ which is defined as

$$\left\{ (r, \theta, t) \mid \frac{L_{i,k}}{2\pi} \leq r \leq \frac{L_{i,k}}{2\pi} + \delta_k, 0 \leq \theta \leq 2\pi, 0 \leq t \leq 2\pi\varepsilon_k \right\} / \sim.$$

Then we have an annulus product $\tilde{Y}_{i,k} = \beta_{i,k}(\bar{Y}_{i,k})$, which is inside of the filled in solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$. In the filling procedure of the Dehn

surgery \mathcal{D}_i , we glue $\tilde{Y}_{i,k}$ with another annulus product $Y_{i,k}$ in $N_{g_k} - T_{i,k}^\circ$ which is defined as

$$\{(r, \theta, t) \mid \varepsilon_k \leq r \leq \varepsilon_k + \delta_k, 0 \leq \theta \leq 2\pi, 0 \leq t \leq L_{i,k}\} / \sim.$$

The gluing is finished by the map $f_{i,k} : \tilde{Y}_{i,k} \rightarrow Y_{i,k}$, which is defined as

$$(r, \theta, t) \mapsto \left(r - \frac{L_{i,k}}{2\pi} + \varepsilon_k, \frac{t}{\varepsilon_k}, \frac{L_{i,k}}{2\pi} \theta \right).$$

We use $\bar{T}_{i,\varepsilon_k}$ to denote the solid torus with radius $L_{i,k}/2\pi$ inside of $\bar{T}_{i,\varepsilon_k+\delta_k}$, and use $\tilde{T}_{i,\varepsilon_k}$ to denote $\beta_{i,k}(\bar{T}_{i,\varepsilon_k})$. If $\bar{m}_{i,k}$ stands for the merid-
ian loop

$$\left\{ \left(\frac{L_{i,k}}{2\pi}, \theta, \pi\varepsilon_k \right) \mid 0 \leq \theta \leq 2\pi \right\}$$

of the boundary torus $\partial\bar{T}_{i,\varepsilon_k}$, then $\tilde{m}_{i,k} = \beta_{i,k}(\bar{m}_{i,k})$ is the meridian loop in the boundary torus $\partial\tilde{T}_{i,\varepsilon_k}$. After gluing by using the filling map $f_{i,k}$, we have $\partial T_{i,\varepsilon_k} = f_{i,k}(\partial\tilde{T}_{i,\varepsilon_k})$. And the meridian loop $\tilde{m}_{i,k}$ will be glued with the longitude loop γ_i on the boundary torus $\partial T_{i,\varepsilon_k}$.

In order to get a smooth Riemannian metric after the Dehn surgery \mathcal{D}_i , we apply the cutoff function technique. We define a smooth cutoff function $\phi_{i,k}$ on $(N_{g_k} - T_{i,\varepsilon_k}^\circ) \cup_{f_{i,k}} \tilde{T}_{i,\varepsilon_k+\delta_k}$ as follows:

$$\phi_{i,k}(x) = \begin{cases} 0, & \text{if } x \in N_{g_k} - (T_{i,\varepsilon_k} \cup Y_{i,k})^\circ; \\ 1, & \text{if } x \in \tilde{T}_{i,\varepsilon_k}. \end{cases}$$

Moreover, we let $\phi_{i,k}(x) \in (0, 1)$ for $x \in Y_{i,k}^\circ$. And for $(\varepsilon_k + \delta_k/2, \theta, t) \in Y_{i,k}^\circ$, with $0 \leq t \leq L_{i,k}$, the value of $\phi_{i,k}$ is assumed to be equal to $1/2$. Then we have a smooth Riemannian metric $\hat{\mathcal{G}}_k$ on

$$(N_{g_k} - T_{i,\varepsilon_k}^\circ) \cup_{f_{i,k}} \tilde{T}_{i,\varepsilon_k+\delta_k}$$

defined as

$$\hat{\mathcal{G}}_k = \begin{cases} \mathcal{G}_k & \text{if } x \in N_{g_k} - (T_{i,\varepsilon_k} \cup Y_{i,k})^\circ; \\ \phi_{i,k}(f_{i,k}^{-1})^* \tilde{\mathcal{G}}_k + (1 - \phi_{i,k}) \mathcal{G}_k & \text{if } x \in Y_{i,k}^\circ = f_{i,k} \left((\tilde{Y}_{i,k})^\circ \right); \\ \tilde{\mathcal{G}}_k & \text{if } x \in \tilde{T}_{i,\varepsilon_k}. \end{cases}$$

For each $i = 1, 2, \dots, n_k$, we use the cutoff function $\phi_{i,k}$ in the Dehn surgery \mathcal{D}_i to get a smooth Riemannian metric. Finally after these n_k Dehn surgeries we have a smooth Riemannian metric defined on N'_{g_k} , still denoted by $\hat{\mathcal{G}}_k$ without any confusion.

Furthermore, the fiber surface Σ_{g_k} has a set of $2g_k$ nonseparating simple geodesic loops $\{\ell_1, \ell_2, \dots, \ell_{2g_k}\}$ representing the basis of $H_1(\Sigma_{g_k}; \mathbb{Z})$. We perform a Dehn surgery around each geodesic loop ℓ_j , with $j = 1, 2, \dots, 2g_k$. All these $2g_k$ Dehn surgeries in N'_{g_k} are performed at the following different regular fiber surface levels:

$$\ell_1 \times \left\{ \frac{5}{6} \right\}, \ell_2 \times \left\{ \frac{5}{6} + \frac{1}{12g_k} \right\}, \dots, \ell_{2g_k} \times \left\{ \frac{5}{6} + \frac{2g_k - 1}{12g_k} \right\}$$

Hence together with Dehn surgeries corresponding to Dehn twists σ_i , all Dehn surgeries are pairwise disjoint when the radius ε_k of removed solid tori is small enough. And compared with Dehn surgeries \mathcal{D}_i corresponding to Dehn twists, the only difference of Dehn surgeries here is that we don't need to twist the filled in solid tori. Other steps are completely the same. After each Dehn surgery, the cutoff function technique is used to obtain a smooth Riemannian metric. After these additional $2g_k$ Dehn surgeries, we have a 3-manifold N''_{g_k} whose double covering manifold is homeomorphic to $S^2 \times S^1$. Hence we have $\pi_1(N''_{g_k}) = \mathbb{Z}_2 * \mathbb{Z}_2$, and then N''_{g_k} is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$. We use $\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}$ to denote N''_{g_k} in the following. The smooth Riemannian metric on $\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}$ obtained through cutoff function technique is still denoted by $\hat{\mathcal{G}}_k$ without any confusion. Therefore we have a sequence of Riemannian 3-manifolds $\{(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathcal{G}}_k)\}_{k=1}^\infty$.

5.2. \mathbb{Z}_2 -coefficient Homology (1, 2)-systolic Freedom. We show that under the sequence of Riemannian metrics $\{\hat{\mathcal{G}}_k\}_{k=1}^\infty$, we have

$$\inf_k \frac{\text{Vol}_{\hat{\mathcal{G}}_k}(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k})}{\text{Sys } H_1(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \cdot \text{Sys } H_2(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2)} = 0. \quad (5.1)$$

The identity (5.1) implies the existence of \mathbb{Z}_2 -coefficient homology (1, 2)-systolic freedom on $\mathbb{R}P^3 \# \mathbb{R}P^3$.

We prove (5.1) by establishing estimations similar to estimations in Theorem 3.2. For \mathbb{Z}_2 -coefficient homology 1-systole, we have the following estimation.

Proposition 5.1. *When k is large enough, the \mathbb{Z}_2 -coefficient homology 1-systole of $(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathcal{G}}_k)$ satisfies*

$$\text{Sys } H_1(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq s_1 (\log g_k)^{1/2}, \quad (5.2)$$

where s_1 is a constant independent of g_k .

Proof. Let γ be a noncontractible loop in $\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}$. If γ crosses a solid torus $\tilde{T}_{i, \varepsilon_k + \delta_k}$ or $\bar{T}_{i, \varepsilon_k + \delta_k}$ in Dehn surgeries, we use a geodesic arc

in the boundary torus $\partial\tilde{T}_{i, \varepsilon_k + \delta_k}$ or $\partial\bar{T}_{i, \varepsilon_k + \delta_k}$ to substitute the arc inside of the solid torus. The substitution geodesic arc in the boundary torus is composed with a geodesic arc parallel to the longitude geodesic loop and another geodesic arc parallel to the meridian geodesic loop. After such a substitution, the increase of the arc length will not exceed the length of meridian geodesic loop of the boundary torus. When we finish the substitutions by considering all possible Dehn surgeries, we have a new loop $\gamma' \subset \mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3$. Assume that k is large enough. The radius ε_k of solid tori would be small enough. Then γ' is homotopic to γ , and thus also noncontractible. We have the following estimation for the length difference,

$$\begin{aligned} |\text{length}_{\hat{\mathcal{G}}_k}(\gamma) - \text{length}_{\hat{\mathcal{G}}_k}(\gamma')| &\leq 2\pi(\varepsilon_k + \delta_k)(2g_k + 1 + 2g_k) \\ &\leq \frac{5\pi}{2}(4g_k + 1)\varepsilon_k. \end{aligned}$$

We have assumed that $\varepsilon_k = 1/g_k^2$. Then when k is large enough, there exists a constant C independent of g_k such that

$$\text{length}_{\hat{\mathcal{G}}_k}(\gamma) \geq \text{length}_{\hat{\mathcal{G}}_k}(\gamma') - C.$$

As γ' has no intersection with solid tori of Dehn surgeries, it is a noncontractible loop in the semibundle N_{g_k} after we do reverse Dehn surgeries on $\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3$. Therefore we have

$$\begin{aligned} \text{length}_{\hat{\mathcal{G}}_k}(\gamma) &\geq \text{length}_{\hat{\mathcal{G}}_k}(\gamma') - C \\ &\geq \text{Sys } \pi_1(N_{g_k}, \mathcal{G}_k) - C \\ &\geq \frac{1}{2} \text{Sys } \pi_1(M_{g_k}, \mathcal{G}_k) - C, \end{aligned}$$

where the last inequality holds because of inequality (4.1) in Proposition 4.3. The proof of Proposition 2.3 in [7] in fact yields that

$$\text{Sys } \pi_1(M_{g_k}, \mathcal{G}_k) \geq C'(\log g_k)^{1/2},$$

where C' is a constant independent of g_k . Then we have

$$\text{length}_{\hat{\mathcal{G}}_k}(\gamma) \geq \frac{1}{2}C'(\log g_k)^{1/2} - C.$$

After taking infimum over all noncontractible loops γ in $\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3$, we have

$$\text{Sys } \pi_1(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k) \geq s_1(\log g_k)^{1/2},$$

where s_1 is a positive constant independent of g_k . Then the inequality (5.2) is implied by the following proposition of systoles:

$$\text{Sys } H_1(\mathbb{R}P^3 \# \mathbb{R}P^3, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq \text{Sys } \pi_1(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k).$$

□

For \mathbb{Z}_2 -coefficient homology 2-systole, we have the following estimation.

Proposition 5.2. *When k is large enough, the \mathbb{Z}_2 -coefficient homology 2-systole of $(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k)$ satisfies*

$$\text{Sys } H_2(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq s_2 g_k, \quad (5.3)$$

where s_2 is a positive constant independent of g_k .

Proof. Assume that X_k is a smooth embedded surface in $\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3$, which is area minimizing among all surfaces representing a nonzero homology class in $H_2(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3; \mathbb{Z}_2)$. The existence of X_k is guaranteed by geometric measure theory. We assume that $((S^2 \times S_{g_k}^1)', \mathcal{G}'_k)$ is the Riemannian covering manifold of $(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k)$. Compared with $S^2 \times S_{g_k}^1$ in M. Freedman's example, the 3-manifold $(S^2 \times S_{g_k}^1)'$ is obtained from M_{g_k} by doing same Dehn surgeries, but the number of Dehn surgeries are doubled. The metric \mathcal{G}'_k on $(S^2 \times S_{g_k}^1)'$ is obtained by applying the cutoff function technique, see Section 3. Hence if we let the radius ε_k of solid tori in Dehn surgeries performed on M_{g_k} be small enough, by applying the same technique in Proposition 2.2 of [7], we have

$$\text{Sys } H_2((S^2 \times S_{g_k}^1)', \mathcal{G}'_k; \mathbb{Z}_2) \geq C g_k, \quad (5.4)$$

where C is a positive constant independent of g_k . On the other hand, the surface X_k can be either lifted to a nonseparating surface \tilde{X}_k in $(S^2 \times S_{g_k}^1)'$ or is doubly covered by a nonseparating surface \tilde{X}_k in $(S^2 \times S_{g_k}^1)'$. Hence we have

$$\begin{aligned} \text{Area}_{\hat{\mathcal{G}}_k}(X_k) &\geq \frac{1}{2} \text{Area}_{\mathcal{G}'_k}(\tilde{X}_k) \\ &\geq \frac{1}{2} \text{Sys } H_2((S^2 \times S_{g_k}^1)', \mathcal{G}'_k; \mathbb{Z}_2) \\ &\geq \frac{1}{2} C g_k. \end{aligned}$$

Then we have

$$\text{Sys } H_2(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geq s_2 g_k,$$

where s_2 is a positive constant independent of g_k . □

We have the following estimation on volume.

Proposition 5.3. *When k is large enough, the volume of*

$$(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k)$$

satisfies

$$\text{Vol}_{\hat{\mathcal{G}}_k}(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3) \leq s_3 g_k, \quad (5.5)$$

where s_3 is a positive constant independent of g_k .

Proof. The same as above, we use $((S^2 \times S_{g_k}^1)', \mathcal{G}'_k)$ to denote the two-sheeted covering manifold of $(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k)$. When k is large enough, the radius ε_k of solid tori in Dehn surgeries on M_{g_k} is small enough, then by applying the same technique in [7], we have

$$\text{Vol}_{\mathcal{G}'_k}((S^2 \times S_{g_k}^1)') \geq C g_k,$$

where C is a positive constant independent of g_k . On the other hand, we have

$$\text{Vol}_{\hat{\mathcal{G}}_k}(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3) = \frac{1}{2} \text{Vol}_{\mathcal{G}'_k}((S^2 \times S_{g_k}^1)').$$

Thus the estimation (5.5) holds. \square

By estimations (5.2), (5.3) and (5.5), we have

$$\begin{aligned} & \frac{\text{Vol}_{\hat{\mathcal{G}}_k}(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3)}{\text{Sys } H_1(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \cdot \text{Sys } H_2(\mathbb{R}P^3 \# \mathbb{R}P_{g_k}^3, \hat{\mathcal{G}}_k; \mathbb{Z}_2)} \\ & \leq \frac{s_3 g_k}{s_1 (\log g_k)^{1/2} \cdot s_2 g_k}. \end{aligned}$$

Therefore, we have (5.1) by letting $k \rightarrow \infty$.

REFERENCES

- [1] I. Agol, The virtual Haken conjecture. With an appendix by Agol, Daniel Groves, and Jason Manning. Doc. Math. 18 (2013), 1045–1087.
- [2] I. Babenko and M. Katz, Systolic freedom of orientable manifolds. Ann. Sci. École Norm. Sup. (4) 31 (1998), no. 6, 787–809.
- [3] I. Babenko and M. Katz and A. Suci, Volumes, middle-dimensional systoles, and Whitehead products. Math. Res. Lett. 5 (1998), no. 4, 461–471.
- [4] L. Bérard-Bergery and M. Katz, On intersystolic inequalities in dimension 3. Geom. Funct. Anal. 4 (1994), no. 6, 621–632.
- [5] P. Buser and P. Sarnak, On the period matrix of a Riemann surface of large genus. With an appendix by J. H. Conway and N. J. A. Sloane. Invent. Math. 117 (1994), no. 1, 27–56.
- [6] B. Farb and D. Margalit, A primer on mapping class groups. Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012.
- [7] M. Freedman, \mathbb{Z}_2 -systolic-freedom. Proceedings of the Kirbyfest (Berkeley, CA, 1998), 113–123 (electronic), Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.

- [8] M. Freedman and D. Meyer and F. Luo, \mathbb{Z}_2 -systolic freedom and quantum codes. Mathematics of quantum computation, Chapman & Hall/CRC (2002): 287–320.
- [9] M. Gromov, Filling Riemannian manifolds. J. Differential Geom. 18 (1983), no. 1, 1–147.
- [10] M. Gromov, Systoles and intersystolic inequalities. Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 291–362, Sémin. Congr., 1, Soc. Math. France, Paris, 1996.
- [11] L. Guth, Metaphors in systolic geometry. Proceedings of the International Congress of Mathematicians. Volume II, 745–768, Hindustan Book Agency, New Delhi, 2010.
- [12] J. Hempel, 3-Manifolds. Reprint of the 1976 original. AMS Chelsea Publishing, Providence, RI, 2004.
- [13] J. Hempel and W. Jaco, Fundamental groups of 3-manifolds which are extensions. Ann. of Math. (2) 95 (1972), 86–98.
- [14] M. Katz, Counterexamples to isosystolic inequalities. Geom. Dedicata 57 (1995), no. 2, 195–206.
- [15] M. Katz, Local calibration of mass and systolic geometry. Geom. Funct. Anal. 12 (2002), no. 3, 598–621.
- [16] M. Katz, Systolic geometry and topology. With an appendix by Jake P. Solomon. Mathematical Surveys and Monographs, 137. American Mathematical Society, Providence, RI, 2007.
- [17] M. Katz and A. Suciu, Volume of Riemannian manifolds, geometric inequalities, and homotopy theory. Tel Aviv Topology Conference: Rothenberg Festschrift (1998), 113–136, Contemp. Math., 231, Amer. Math. Soc., Providence, RI, 1999.
- [18] M. Katz and A. Suciu, Systolic freedom of loop space. Geom. Funct. Anal. 11 (2001), no. 1, 60–73.
- [19] C. Pittet, Systoles on $S^1 \times S^n$. Differential Geom. Appl. 7 (1997), no. 2, 139–142.
- [20] P. Pu, Some inequalities in certain nonorientable Riemannian manifolds. Pacific J. Math. 2, (1952). 55–71.
- [21] D. Rolfsen, Knots and Links, Providence (R.I.) : American mathematical society, 2003.
- [22] P. Schmutz Schaller Extremal Riemann surfaces with a large number of systoles. Extremal Riemann surfaces (San Francisco, CA, 1995), 9–19, Contemp. Math., 201, Amer. Math. Soc., Providence, RI, 1997.
- [23] W. Thurston, Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle. arXiv preprint math/9801045.
- [24] L. Zulli, Semibundle decompositions of 3-manifolds and the twisted cofundamental group. Topology Appl. 79 (1997), no. 2, 159–172.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY
STILLWATER, OKLAHOMA 74078-0613
E-mail address: lizhi@ostatemail.okstate.edu